Vacuum energy in the presence of a magnetic string with a delta function profile

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We present a calculation of the ground state energy of massive spinor fields and massive scalar fields in the background of an inhomogeneous magnetic string with a potential given by a delta function. The zeta functional regularization is used and the lowest heat kernel coefficients are calculated. The rest of the analytical calculation adopts the Jost function formalism. In the numerical part of the work the renormalized vacuum energy as a function of the radius R of the string is calculated and plotted for various values of the strength of the potential. The sign of the energy is found to change with the radius. For both scalar and spinor fields the renormalized energy shows no logarithmic behavior in the limit $R\to 0$, as was expected from the vanishing of the heat kernel coefficient A_2, which is not zero for other types of profiles.

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I. INTRODUCTION

Recent developments in the technique for the calculation of the zero point energy of massive fields [1] have opened up interesting possibilities in the study of soft boundaries and smooth potentials with spherical and cylindrical geometry immersed in the vacuum of various quantum fields. The technique uses the Jost function of the scattering problem related to the background potential under examination. The Jost function is unique and easily obtainable for many types of geometries with circular, spherical, or cylindrical symmetry and for various types of potential profiles, which is the great advantage of this approach. The renormalization is then performed with zeta-functional techniques and by means of the heat kernel expansion, which is known to be a very effective tool in this context (see [2–4]). The basic idea of defining unambiguously a renormalized zero point energy, is to impose the normalization condition that the vacuum energy vanishes when the mass of the quantum field reaches infinity (see [1]). The heat kernel coefficients themselves are also of great interest; they determine the asymptotic behavior of the renormalized energy [5]; and they are considered to be an intrinsic feature of the background under examination.

This renormalization scheme and the representation of the energy in terms of the Jost function have been used to solve some basic configurations [6,1,7,8,5,9]. The investigation has turned recently to magnetic fields with cylindrical symmetry. In [10] a complete analysis of a spinor field in the background of a homogeneous magnetic flux tube of finite radius was carried out. The vacuum energy was found to be negative and it did not show a minimum for any finite value of the radius. A natural question is if inhomogeneous magnetic fields can minimize the energy and render the string stable. The question was already raised in [11]. The present paper extends the investigation begun in [10] to an inhomogeneous magnetic string with a delta function profile. The delta function, although not a fully realistic physical model, represents a simple example of inhomogeneity, which could give an insight into the problem of vacuum energy in magnetic backgrounds. This kind of ''semitransparent'' boundary was already analyzed in [7] for a sphere. It has some features in common with a smooth potential and some with a hard boundary. In [12] the heat kernel coefficients for a general semitransparent boundary were calculated. The quantum mechanics of spinor fields in the presence of magnetic fluxes has been elaborated in early works [13,14] while a more recent investigation in this direction has been motivated by the interest in the Aharonov–Bohm effect [15,16]. Singular inhomogeneous magnetic fields were examined in [17] for the calculation of the fermion determinant and in [18] for the investigation of the bound states of an electron, however the ground state energy was not calculated in those works. In our paper the ground state energy will be calculated for a scalar and for a spinor field. In the first part of this paper we will calculate the Jost functions for both fields and the heat kernel coefficients will be found. In the second part of the paper we will work numerically on the renormalized energy, finding its asymptotic behavior for small and large values of the radius of the string. We will finally show some plots of the renormalized energy for various values of the strength of the potential.

II. SCALAR FIELD IN THE BACKGROUND OF A MAGNETIC STRING

A. Solution of the field equation

We quantize a scalar field Φ in the presence of a classical magnetic field whose form is that of a cylindrical shell with delta function profile. The section of the string is a circle with radius R. The magnetic field is given by

$$\vec{B}(r)=\frac{\phi}{2\pi R}\,\delta(r-R)\vec{e}_z, \tag{1}$$

where ϕ is the magnetic flux, $r=\sqrt{x^2+y^2}$, and z is the axis along which the cylindrical shell extends to infinity. The quantum field obeys the Klein–Gordon equation for the scalar electrodynamics

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$$(D^\mu D_\mu + m_e^2)\Phi(x) = 0, \tag{2}$$

where m_e is the mass of the field, and $D_\mu = \partial_\mu - ieA_\mu$. The vector potential of the electromagnetic field associated with Eq. (1) contains a theta function

$$\vec{A} = \frac{\phi}{2\pi}\frac{\Theta(r-R)}{r}\vec{e}_\varphi, \quad A^0 = 0. \tag{3}$$

Then, after separation of the variables, the field equation in cylindrical coordinates reads

$$\left(k^2 - \frac{[m-\beta\Theta(r-R)]^2}{r^2} + \frac{1}{r}\partial_r + \partial_r^2\right)\Phi_m(k,r) = 0. \tag{4}$$

Here m is the orbital momentum quantum number and $k = \sqrt{p_0^2 - m_e^2 - p_z^2}$, where p_μ is the momentum four vector. The variable $\beta = e\phi/2\pi$ represents the strength of the background potential. The solutions to Eq. (4) are Bessel and Neumann functions. The kind of function and their coefficients are determined by means of physical considerations. We take here the regular solution which is known from the general scattering theory [19]

$$\Phi(r) = J_m(kr)\Theta(R-r) + \tfrac{1}{2}[f_m(k)H^{(2)}_{m-\beta}(kr) + f_m^*(k)H^{(1)}_{m-\beta}(kr)]\Theta(r-R), \tag{5}$$

where $J_m(kr)$ is a Bessel function of the first kind, $H^{(1)}_{m-\beta}(kr)$ and $H^{(2)}_{m-\beta}(kr)$ are Hankel functions of the first and second kind, and the coefficients $f_m(k)$ and $f_m^*(k)$ are a Jost function and its complex conjugate, respectively. Then, we can define a field Φ^I in the region $r<R$ inside the cylinder

$$\Phi^I_m(k,r) = J_m(kr), \tag{6}$$

which is independent of the strength β of the potential, and a field Φ^O in the region outside the cylinder $r>R$

$$\Phi^O_m(k,r) = \tfrac{1}{2}[f_m(k)H^{(2)}_{m-\beta}(kr) + f_m^*(k)H^{(1)}_{m-\beta}(kr)], \tag{7}$$

which describes incoming and outgoing cylindrical waves. The conditions for the field at $r=R$ will be discussed later.

B. Ground state energy in terms of the Jost function and normalization condition

A regularized vacuum energy can be defined as

$$E_0^{\rm sc} = \frac{\mu^{2s}}{2}\sum \epsilon_{(n,\alpha)}^{1-2s}, \tag{8}$$

where the $\epsilon_{(n,\alpha)}$ are the eigenvalues of the Hamiltonian operator associated with Eq. (2), $\alpha = \pm 1$ being the index for the particle-antiparticle degree of freedom, while n includes all other quantum numbers. s is the regularization parameter to be put to zero after the renormalization and μ is a mass parameter necessary to maintain the correct dimensions of the energy. The string is invariant under translations along the z axis, therefore the energy density per unit length is

$$\mathcal{E}^{\rm sc} = \frac{1}{2}\mu^{2s}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{(n,\alpha)}(p_z^2 + \lambda_{(n)}^2)^{1/2-s}, \tag{9}$$

where the $\lambda_{(n)}$ are the eigenvalues of the operator contained in Eq. (4) with $k = \sqrt{p_0^2 - m_e^2}$. We perform the integration over p_z in Eq. (9), getting

$$\mathcal{E}^{\rm sc} = \frac{1}{4}\mu^{2s}\frac{\Gamma(s-1)}{\sqrt{\pi}\Gamma(s-1/2)}\sum_{(n,\alpha)}(\lambda_{(n)}^2)^{1-s}. \tag{10}$$

Now, following a known procedure [1], the sum in Eq. (10) can be transformed into an integral containing the Jost function introduced in Eq. (5):

$$\mathcal{E}^{\rm sc} = -\frac{1}{2}C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2 - m_e^2)^{1-s}\partial_k \ln f_m(ik), \tag{11}$$

where $f_m(ik)$ is the Jost function with imaginary argument and $C_s = \{1 + s[-1 + 2\ln(2\mu)]\}/(2\pi)$ is a simple function of the regularization parameter. The renormalization of Eq. (11) is carried out by direct subtraction of its divergent part

$$\mathcal{E}^{\rm sc}_{\rm ren} = \mathcal{E}^{\rm sc} - \mathcal{E}^{\rm sc}_{\rm div}. \tag{12}$$

The isolation of $\mathcal{E}^{\rm sc}_{\rm div}$ will be performed via heat-kernel expansion as we will see in a moment. The subtracted part should be added in the classical part of the energy resulting in a renormalization of the classical parameters of the string (in [8] this procedure is well explained), however we do not treat the classical energy of the system here but only the vacuum contribution. For the analytical continuation $s \to 0$, we split $\mathcal{E}^{\rm sc}_{\rm ren}$ into a "finite" and an "asymptotic" part:

$$\mathcal{E}^{\rm sc}_{\rm ren} = \mathcal{E}^{\rm sc}_f + \mathcal{E}^{\rm sc}_{\rm as}, \tag{13}$$

with

$$\mathcal{E}^{\rm sc}_f = -\frac{1}{2}C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2 - m_e^2)^{1-s}\frac{\partial}{\partial k}\times[\ln f_m(ik) - \ln f_m^{\rm as}(ik)] \tag{14}$$

and

$$\mathcal{E}^{\rm sc}_{\rm as} = -\frac{1}{2}C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2 - m_e^2)^{1-s}\frac{\partial}{\partial k}\ln f_m^{\rm as}(ik) - \mathcal{E}^{\rm sc}_{\rm div}, \tag{15}$$

where $f_m^{\rm as}$ is a portion of the uniform asymptotic expansion of the Jost function. The number of orders to be included in $f_m^{\rm as}$ must be sufficient to let the function

$$\ln f_m(ik) - \ln f_m^{\rm as}(ik) \tag{16}$$

fall as m^{-4} (or k^{-4}) for k and m equally large, in this case the integral and the summation in Eq. (14) converge for $s \to 0$. To this purpose three orders in the asymptotics are enough. More orders would only give a quicker convergence. The splitting proposed in Eq. (13) immediately permits the analytical continuation $s=0$ in $\mathcal{E}_f^{\rm sc}$, furthermore it allows a very quick subtraction, of the pole terms in the asymptotic part Eq. (15). $\mathcal{E}_{\rm as}^{\rm sc}$ is a finite quantity at $s=0$. For the definition of $\mathcal{E}_{\rm div}^{\rm sc}$ we expand Eq. (10) in powers of the mass by means of the heat-kernel coefficients A_j associated with the Hamilton operator

$$\mathcal{E}^{\rm sc}=\sum_j \frac{\mu^{2s}}{32\pi^2}\frac{\Gamma(s+j-2)}{\Gamma(s+1)}m_e^{4-2(s+j)}A_j, \quad j=0,\frac{1}{2},1,\ldots. \tag{17}$$

Here the divergent contribution can be isolated

$$\begin{aligned}\mathcal{E}_{\rm div}^{\rm sc}=&-\frac{m_e^4}{64\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m_e^2}-\frac{1}{2}\right)A_0-\frac{m_e^3}{24\pi^{3/2}}A_{1/2}\\&+\frac{m_e^2}{32\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m_e^2}-1\right)A_1+\frac{m_e}{16\pi^{3/2}}A_{3/2}\\&-\frac{1}{32\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m_e^2}-2\right)A_2.\end{aligned} \tag{18}$$

In this definition the poles are all contained in the three terms corresponding to the heat kernel coefficients A_0, A_1, A_2, however we included in $\mathcal{E}_{\rm div}^{\rm sc}$ two more terms in order to satisfy a normalization condition, namely that the renormalized ground state energy vanishes for a field of infinite mass

$$\lim_{m_e\to\infty}\mathcal{E}_{\rm ren}^{\rm sc}=0. \tag{19}$$

This condition fixes a unique value for the vacuum energy of a massive field. It is also necessary to eliminate the arbitrariness of the mass parameter μ.

C. The Jost function and its asymptotics

We now calculate the Jost function $f_m(k)$ in relation to the delta function potential of the magnetic string. To this aim we study the matching conditions for the field at $r=R$. We require that the field be continuous on the boundary. From this condition and from the field equation (4) it follows that

$$\begin{aligned}\Phi^0(r)|_{r=R}&=\Phi^I(r)|_{r=R},\\ \partial_r\Phi^O(r)|_{r=R}&=\partial_r\Phi^I(r)|_{r=R}.\end{aligned} \tag{20}$$

Inserting in this system the solutions Eqs. (6) and (7) and solving for $f_m(k)$ it is possible to find the Jost function on the imaginary axis.[1] We write it in terms of modified Bessel I and K functions

$$f_m(ik)=i^\beta kR(I_mK_{m-\beta+1}+I_{m+1}K_{m-\beta})+i^\beta\beta I_mK_{m-\beta}. \tag{21}$$

[1] The Wronskian determinant for the Hankel functions [20] is used.

In Eq. (21) the arguments (kR) of the Bessel functions are omitted for simplicity. This expression holds for positive and for negative values of m. On the contrary the uniform asymptotic expansion, which we need for formulas (14) and (15), is a different function for positive or for negative m. To find it, we adopt the asymptotic expansion of the modified Bessel I and K functions for large indices and large arguments available on [20]. Our calculations need an expansion of the form

$$K_{m+\alpha}(kR)\sim\sum_n\frac{X_n}{m^n}, \tag{22}$$

where α can be 0 or 1 for the Bessel I function and $-\beta$ or $-\beta+1$ for the Bessel K function and the X_n are some coefficients depending on k, R, and β. Therefore we reexpand the formulas given in [20] in powers of m finding

$$\begin{aligned}K_{m+\alpha}(kR)&\sim\sqrt{\frac{\pi}{2m}}\exp\left\{\sum_{n=-1}^{3}m^{-n}S_\eta(n,\alpha,t)\right\},\\ I_{m+\alpha}(kR)&\sim\frac{1}{\sqrt{2\pi m}}\exp\left\{\sum_{n=-1}^{3}m^{-n}S_\eta(n,\alpha,t)\right\},\end{aligned} \tag{23}$$

where $t=[1+(kR/m)^2]^{-1/2}$ and the functions $S_\eta(n,\alpha,t)$ are given explicitly in the Appendix. Inserting these expansions in Eq. (21) one finds an asymptotic Jost function valid for positive m; we name it $f_m^{\rm as+}(ik)$.

To find the asymptotics for negative m we must invert the sign of the indices of the Bessel functions in Eq. (21). This operation does not change the Jost function at all, because of the remarkable algebraic properties of the modified Bessel functions. After inverting the signs we can apply expansions Eq. (23) and find the desired result. We call this contribution $f_m^{\rm as-}(ik)$. For the term $m=0$, formula (22) clearly does not apply. One obtains the contribution $f_0^{\rm as}(ik)$ from the commonly known expansions of the modified Bessel functions for large arguments available in [20].

The finite and the asymptotic part of the energy defined in Eqs. (14) and (15) are also split into three contributions: one for positive m, one for negative m, and one for $m=0$. The positive and negative contributions can be summed up in a single term, but the contribution coming from $m=0$ must be calculated separately and summed just numerically at the end, in fact we have

$$\begin{aligned}\mathcal{E}_f^{\rm sc}=-\frac{1}{2}C_s\Bigg(&\sum_{m=1}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\partial_k[\ln f_m^{\pm}(ik)\\&-\ln f_m^{\rm as\pm}(ik)]+\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\partial_k[\ln f_0(ik)\\&-\ln f_0^{\rm as}(ik)]\Bigg),\end{aligned} \tag{24}$$

$$\mathcal{E}^{\rm sc}_{\rm as}=-\frac{1}{2}C_s\left(\sum_{m=1}^{\infty} dk(k^2-m_e^2)^{1-s}\partial_k \ln f^{{\rm as}\pm}_m(ik)\right.$$
$$\left.+\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\partial_k \ln f^{\rm as}_0(ik)-\mathcal{E}^{\rm sc}_{\rm div}\right), \tag{25}$$

where $f_m^{\pm}=f_m(ik)+f_{-m}(ik)$, and $\ln f^{{\rm as}\pm}_m(ik)=\ln f^{{\rm as}+}_m(ik)+\ln f^{{\rm as}-}_{-m}(ik)$.

Taking the logarithm of $f^{{\rm as}+}_m(ik)$ and $f^{as-}_{-m}(ik)$ and reexpanding in powers of m we finally find the explicit expression for the required functions up to the third order:

$$\ln f^{\rm as}_0=\frac{\beta^2}{2kR};\quad \ln f^{{\rm as}\pm}_m(ik)=\sum_{n=1}^{3}\sum_{t} X_{n,j}\frac{t^j}{m^n}, \tag{26}$$

where the nonzero coefficients are

$$X_{1,1}=\beta^2,\quad X_{2,4}=\beta^2/4,\quad X_{3,3}=\beta^2/24-\beta_4/12,$$
$$X_{3,5}=-\beta^2/2+\beta^4/4,\quad X_{3,7}=\beta^4/16. \tag{27}$$

As we mentioned above three orders in m are sufficient[2] for the convergence of $\mathcal{E}^{\rm sc}_f$.

D. The asymptotic part of the energy and the heat kernel coefficients

Having found the Jost function related to the cylindrical delta potential an important part of the calculation is done. We proceed with the analytical simplification of $\mathcal{E}^{\rm sc}_{\rm as}$. The second term in Eq. (25), which we name $\mathcal{E}^{\rm sc}_{\rm as0}$, can be quickly calculated

$$\mathcal{E}^{\rm sc}_{\rm as0}=-\frac{\beta^2 m_e}{4\pi R}. \tag{28}$$

The first term in Eq. (25), which we name here $\mathcal{E}^{\rm sc}_{{\rm as}(m)}$, can be transformed with the Abel–Plana formula

$$\sum_{m=1}^{\infty}F(m)=\int_0^{\infty}dmF(m)-\frac{1}{2}F(0)$$
$$+\int_0^{\infty}\frac{dm}{1-e^{2\pi m}}\frac{F(im)-F(-im)}{i}. \tag{29}$$

In our case the function $F(m)$ is

$$F(m)=\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\partial_k \ln f^{{\rm as}\pm}_m(ik). \tag{30}$$

Thus $\mathcal{E}^{\rm sc}_{{\rm as}(m)}$ is split into three terms: one for each of the addends on the right hand side of Eq. (29). We call these terms $\mathcal{E}^{\rm sc}_{\rm as1}$, $\mathcal{E}^{\rm sc}_{\rm as2}$, and $\mathcal{E}^{\rm sc}_{\rm as3}$. The contributions $\mathcal{E}^{\rm sc}_{\rm as1}$ and $\mathcal{E}^{\rm sc}_{\rm as2}$ can be calculated with the formulas given in Appendix B. The result is

$$\mathcal{E}^{\rm sc}_{\rm as1}=\frac{\beta^2 m_e^2}{8\pi}\left[\frac{1}{s}+\ln\left(\frac{4\mu^2}{m_e^2}\right)-1\right]-\frac{\beta^2 m_e}{32R}, \tag{31}$$

$$\mathcal{E}^{\rm sc}_{\rm as2}=\frac{\beta^2 m_e}{4\pi R}-\frac{\beta^2}{96\pi m_e R^3}+\frac{\beta^4}{48\pi m_e R^3}. \tag{32}$$

The divergences are all contained in $\mathcal{E}^{\rm sc}_{\rm as1}$. The first addend of $\mathcal{E}^{\rm sc}_{\rm as2}$ cancels with $\mathcal{E}^{\rm sc}_{\rm as0}$, and we are left with only one term containing a positive power of the mass. This term is the contribution to $\mathcal{E}^{\rm div}_{\rm sc}$ corresponding to the heat-kernel coefficient $A_{3/2}$ [see Eq. (18)] and therefore it will be subtracted as well as the pole term. The last term of the Abel–Plana formula demands a little more work. Using the formula displayed in Appendix B to calculate the integral over k, we find

$$\mathcal{E}^{\rm sc}_{\rm as3}=-\frac{\beta^2}{\pi R^2}h_1(m_eR)+\left(\frac{\beta^2}{24\pi R^2}-\frac{\beta^4}{12\pi R^2}\right)h_2(m_eR)$$
$$+\left(-\frac{\beta^2}{6\pi R^2}+\frac{\beta^4}{12\pi R^2}\right)h_3(m_eR)$$
$$+\frac{\beta^2}{24\pi R^2}h_4(m_eR)+\frac{\beta^2 m_e}{16R}\frac{1}{1-e^{2\pi m_e R}}. \tag{33}$$

The functions $h_n(x)$ are given by

$$h_1(x)=\int_x^{\infty}\frac{dm}{1-e^{2\pi m}}\sqrt{m^2-x^2},$$
$$h_2(x)=\int_x^{\infty}dm\left(\frac{1}{1-e^{2\pi m}}\frac{1}{m}\right)'\sqrt{m^2-x^2},$$
$$h_3(x)=\int_x^{\infty}dm\left[\left(\frac{m}{1-e^{2\pi m}}\right)'\frac{1}{m}\right]'\sqrt{m^2-x^2}, \tag{34}$$
$$h_4(x)=\int_x^{\infty}dm\left\{\left[\left(\frac{m^3}{1-e^{2\pi m}}\right)'\frac{1}{m}\right]'\frac{1}{m}\right\}'\sqrt{m^2-x^2},$$

where the prime in the integrands denotes the derivative with respect to m. The heat-kernel coefficients, which we have calculated up to the coefficient $A_{7/2}$ (including four more orders in $\ln f^{{\rm as}\pm}(ik)$), read

$$A_0=0,\quad A_{1/2}=0,$$

$$A_1=4\pi\beta^2,\quad A_{3/2}=\frac{\beta^2\pi^{3/2}}{2R},$$

$$A_2=0,\quad A_{5/2}=\frac{[(3\pi-128)\beta^2+(256-18\pi)\beta^4]\pi^{1/2}}{384R^3},$$

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[2] We would like to stress that with the introduction of asymptotic expansions in our calculation we do not approximate the vacuum energy. The total energy as defined in Eq. (12) remains an exact quantity.

$$A_3=0, \quad A_{7/2}=\frac{(27\beta^2-100\beta^4+80\beta^6)\pi^{3/2}}{24\,576R^5}. \tag{35}$$

We perform the subtraction proposed in Eq. (12) and we obtain the final result

$$\begin{aligned}\mathcal{E}^{\rm sc}_{\rm as}=&-\frac{\beta^2}{\pi R^2}h_1(m_eR)+\left(\frac{\beta^2}{24\pi R^2}-\frac{\beta^4}{12\pi R^2}\right)h_2(m_eR)\\&+\left(\frac{-\beta^2}{6\pi R^2}+\frac{\beta^4}{12\pi R^2}\right)h_3(m_eR)+\frac{\beta^2}{24\pi R^2}h_4(m_eR)\\&-\frac{\beta^2}{96\pi m_eR^3}+\frac{\beta^4}{48\pi m_eR_3}+\frac{\beta^2m_e}{16R}\frac{1}{1-e^{2\pi m_eR}}.\end{aligned}\tag{36}$$

The functions $h_n(x)$ are convergent integrals which can be easily calculated numerically.

The finite part of the ground state energy given by Eq. (14) can be integrated by parts giving

$$\begin{aligned}\mathcal{E}^{\rm sc}_f=&\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{m_e}^{\infty}dkk\left[\ln f_m^{\pm}(ik)-\sum_n^3\sum_t X_{n,j}\frac{t^j}{m^n}\right]\\&+\frac{1}{2\pi}\int_{m_e}^{\infty}dkk\left[\ln f_0(ik)-\frac{\beta^2}{2kR}\right].\end{aligned}\tag{37}$$

Here we name the first addend $\mathcal{E}^{\rm sc}_{fm}$ and the second addend $\mathcal{E}^{\rm sc}_{f0}$; in the plots we will display them separately. Equations (36) and (37) are considered the main analytical result of this paper concerning the scalar field. Their sum gives the total renormalized vacuum energy. The sum will be performed in the numerical part of this paper.

III. SPINOR FIELD IN THE BACKGROUND OF A MAGNETIC STRING

A. Solution of the field equation

An analysis of a spinor field in the background of a cylindrical magnetic field with an arbitrary profile has been performed in [10]. The field equation for a spinor with components $g_1(r)$ and $g_2(r)$ in the background of a translationally invariant potential with delta function profile is

$$\begin{pmatrix} p_0-m_e & \partial_r-\dfrac{m-\beta\Theta(R-r)}{r}\\ -\partial_r-\dfrac{m+1-\beta\Theta(R-r)}{r} & p_0+m_e\end{pmatrix}\times\begin{pmatrix}g_1(r)\\ g_2(r)\end{pmatrix}=0. \tag{38}$$

The reader is referred to [10] for a derivation of this equation. Let us find the solution to Eq. (38) for one component of the spinor. The decoupled equation for the component g_2 is

$$\left(k^2-\frac{[m-\beta\Theta(R-r)]^2}{r^2}+\frac{\beta}{r}\delta(R-r)+\frac{1}{r}\partial_r+\partial_r^2\right)g_2(r)=0, \tag{39}$$

where $k=\sqrt{p_0^2-m_e^2}$. The regular solution in the region $r<R$ is

$$g_2^I(r)=J_m(kr) \tag{40}$$

and in the region $r>R$

$$g_2^O(r)=\tfrac{1}{2}[f_m^{\rm spin}(k)H^{(2)}_{m-\beta}(kr)+f_m^{{\rm spin}*}(k)H^{(1)}_{m-\beta}(kr)]. \tag{41}$$

Here $f_m^{\rm spin}(k)$ and $f_m^{{\rm spin}*}(k)$ are the Jost function and its conjugate related to the scattering problem for the spinor field.

The ground state energy of the spinor field in the background of the magnetic string is

$$E_0=-\frac{\mu^2}{2}\sum_{n,\alpha,\sigma}\epsilon^{1-2s}_{(n,\alpha,\sigma)}, \tag{42}$$

where the minus sign accounts for the change of the statistics, and the $\epsilon_{(n,\alpha,\sigma)}$ are the eigenvalues of the Hamiltonian

$$H=-i\gamma^0\gamma^l[\partial_{x^l}-ieA_l(x)]+\gamma^0m_e. \tag{43}$$

The degree of freedom σ accounts for the two independent spin states. As in the scalar case we calculate the energy for a section of the string. The ground state energy density per unit length of the string in terms of the Jost function is given by

$$\mathcal{E}^{\rm spin}=C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\partial_k\ln f_m^{\rm spin}(ik). \tag{44}$$

The renormalization scheme is the same we introduced for the scalar case. The expansion of the ground state energy in powers of the mass and the definition of $\mathcal{E}^{\rm spin}_{\rm div}$ are the same as in Eqs. (17) and (18), apart from a factor -1 coming from the change of the statistics. The heat kernel coefficients will be of course not the same, we call them B_n. The normalization condition Eq. (19) remains unchanged. The ground state energy is split into two parts: $\mathcal{E}^{\rm spin}_f$ and $\mathcal{E}^{\rm spin}_{\rm as}$, much in the same way as in Eqs. (14) and (15), with the introduction of the uniform asymptotic expansion of the Jost function $f_m^{\text{spin-as}}(ik)$, which we calculate in the next subsection.

B. Matching conditions and Jost function

We require the continuity of the field on the surface of the flux tube. This, together with Eq. (38), produce the following matching conditions at $r=R$:

$$\begin{aligned}&g_2^I(r)|_{r=R}=g_2^O(r)|_{r=R},\\&[\partial_rg_2^I(r)]|_{r=R}-[\partial_rg_2^O(r)]|_{r=R}=\frac{\beta}{r}g_2^I(r)|_{r=R}.\end{aligned}\tag{45}$$

Inserting in Eq. (45) solutions (40) and (41), and solving for $f_m(k)^{\rm spin}$, we find the desired result. It turned out to be more convienent to express the Jost function with the parameter ν which is given by

$$\nu=\begin{cases} m+1/2 & \text{for } m=0,1,2,..., \\ -m-1/2 & \text{for } m=-1,-2,..., \end{cases} \tag{46}$$

with $\nu=\frac{1}{2},\frac{3}{2},\ldots$ in both cases. Then the Jost functions on the imaginary axis read

$$f_\nu^+(ik)=i^\beta kR(I_{\nu+1/2}K_{\nu-1/2-\beta}+I_{\nu-1/2}K_{\nu+1/2-\beta}), \quad m\geqslant 0, \tag{47}$$

which can be expanded for large positive m and

$$f_\nu^-(ik)=i^{-\beta} kR(I_{\nu+1/2}K_{\nu-1/2+\beta}+I_{\nu-1/2}K_{\nu+1/2+\beta}), \quad m<0, \tag{48}$$

which can be expanded for large negative m. The asymptotic expansions of the Bessel I and K functions for ν and k equally large are obtained with formula (23). From these formulas the logarithm of the asymptotic Jost function can be easily calculated up to the third order and we define

$$\ln f_\nu^{\text{as-spin}}(ik)=\sum_{n,j}^{3} Y_{n,j}\frac{t^j}{\nu^n}, \tag{49}$$

where $t=1/(1+(kR/\nu)^2)^{1/2}$ and the nonzero coefficients are

$$Y_{1,1}=\beta^2, \quad Y_{2,2}=-\beta^2/4, \quad Y_{2,4}=\beta^2/4,$$
$$Y_{3,3}=\beta^2/6-\beta^4/12,$$
$$Y_{3,5}=-7\beta^2/8+\beta^4/4, \quad Y_{3,7}=5\beta^2/8. \tag{50}$$

C. The asymptotic and the finite part of the energy

The asymptotic part of the energy can be written, using result Eq. (49), as

$$\mathcal{E}_{\rm as}^{\rm spin}=C_s\sum_{\nu=1/2}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\frac{\partial}{\partial k}\sum_{j,n}^{3} Y_{j,n}\frac{t^j}{\nu^n}-\mathcal{E}_{\rm div}^{\rm spin}, \tag{51}$$

we calculate the sum over ν with the help of the Abel–Plana formula for half integer variables which can be found in the Appendix. The case $m=0$ (i.e., $\nu=1/2$) does not need to be treated separately. We have only the two contributions

$$\mathcal{E}_{\rm as1}^{\rm spin}=\frac{\beta^2 m_e^2}{4\pi}\left[\frac{1}{s}+\ln\left(\frac{4\mu^2}{m_e^2}\right)-1\right]-\frac{\beta^2 m_e}{16R} \tag{52}$$

and

$$\mathcal{E}_{\rm as2}^{\rm spin}=\frac{2\beta^2}{\pi R^2}q_1(m_R)+\left(-\frac{\beta^2}{3\pi R^2}+\frac{\beta^4}{6\pi R^2}\right)q_2(m_eR)+\left(\frac{7\beta^2}{12\pi R^2}+\frac{\beta^4}{6\pi R^2}\right)q_3(m_eR)-\frac{\beta^2}{12\pi R^2}q_4(m_eR), \tag{53}$$

where the functions $q_n(x)$ are

$$q_1(x)=\int_x^\infty\frac{d\nu}{1+e^{2\pi\nu}}\sqrt{\nu^2-x^2},$$
$$q_2(x)=\int_x^\infty d\nu\left(\frac{1}{1+e^{2\pi\nu}}\frac{1}{\nu}\right)'\sqrt{\nu^2-x^2},$$
$$q_3(x)=\int_x^\infty d\nu\left[\left(\frac{\nu}{1+e^{2\pi\nu}}\right)'\frac{1}{\nu}\right]'\sqrt{\nu^2-x^2}$$
$$q_4(x)=\int_x^\infty d\nu\left\{\left[\left(\frac{\nu^3}{1+e^{2\pi\nu}}\right)'\frac{1}{\nu}\right]'\frac{1}{\nu}\right\}'\sqrt{\nu^2-x^2}. \tag{54}$$

The only pole term is contained in $\mathcal{E}_{\rm as1}^{\rm spin}$ and the term proportional to m_e will be subtracted as well as the pole term, thus Eq. (52) cancels completely with the subtraction of $\mathcal{E}_{\rm div}^{\rm spin}$. Therefore we finally have $\mathcal{E}_{\rm as}^{\rm spin}=\mathcal{E}_{\rm as2}^{\rm spin}$. The heat-kernel coefficients, which we have calculated up to the coefficient B_4, read

$$B_0=0, \quad B_{1/2}=0,$$
$$B_1=8\pi\beta^2, \quad B_{3/2}=-\beta^2\pi^{3/2}/R,$$
$$B_2=0, \quad B_{5/2}=\frac{(3\beta^2+2\beta^4)\pi^{3/2}}{64R^3},$$
$$B_3=0, \quad B_{7/2}=-\frac{(135\beta^2-68\beta^4+16\beta^6)\pi^{3/2}}{12288R^5},$$
$$B_4=\frac{5\beta^8}{1281R^6}. \tag{55}$$

It is interesting to note how in both scalar and spinor cases we found the final expression for the asymptotic part of the energy to depend only on even powers of β. This was to be expected for physical reasons, in fact by inverting the direction of the magnetic flux ϕ the ground state energy should not change.

The finite part of the energy $\mathcal{E}_f^{\rm spin}$ can be hardly analytically simplified. We can only integrate by parts to obtain a final form which is suitable for the numerical calculation

$$\mathcal{E}_f^{\rm spin}=-\frac{1}{\pi}\sum_{\nu=1/2}^{\infty}\int_{m_e}^{\infty}dkk\left[\ln f_\nu^\pm(ik)-\sum_{n,j}^{3,7} Y_{n,j}\frac{t^j}{\nu^n}\right], \tag{56}$$

where $f_\nu^\pm(ik)=f_\nu^+(ik)+f_\nu^-(ik)$.

IV. NUMERICAL EVALUATIONS

In this section we show some graphics of $\mathcal{E}_{as},\mathcal{E}_f$ and of the complete renormalized vacuum energy $\mathcal{E}_{ren}$ as a function of the radius of the string, for the scalar and for the spinor field. We calculate also the asymptotic behavior of $\mathcal{E}_{as}$ and $\mathcal{E}_f$ for large and for small R. Since we want to study here only the dependence on R and on β, we set $m_e=1$ for all the calculations of this section.

A. Scalar field

As a first step we rewrite $\mathcal{E}_{as}^{sc}$ in a form in which the dependence on the relevant parameters is more explicit:

$$\mathcal{E}_{as}^{sc}=\frac{1}{\pi R^2}[\beta^2 g_1(m_e R)+\beta^4 g_2(m_e R)], \tag{57}$$

where the functions $g_1(x)$ and $g_1(x)$ are linear combinations of the integrals given in Eq. (34). The asymptotic behavior of these functions for $x\to 0$ is

$$g_1(x)\sim 0.0317+\mathcal{O}(x),$$

$$g_2(x)\sim 0.0417+\mathcal{O}(x). \tag{58}$$

The logarithmic contributions have canceled. This was to be expected from the vanishing of the heat-kernel coefficient A_2. The contribution $\mathcal{E}_{as}^{sc}$ is proportional to R^{-2} for $R\to 0$. For $R\to\infty$ all the $h_n(x)$ functions fall exponentially and so does $\mathcal{E}_{as}^{sc}$.

The finite part $\mathcal{E}_f^{sc}$ is also proportional to R^{-2} in the limit $R\to 0$. For large R we found numerically $\mathcal{E}_f^{sc}\sim R^{-3}$, which is in agreement with the heat-kernel coefficient $A_{5/2}$ shown in Eq. (35), in fact the first nonvanishing heat-kernel coefficient after A_2 determines the behavior of the renormalized energy for $R\to\infty$. Below we show the plots of all the contributions to the renormalized ground state energy. Each contribution has been multiplied by R^2 so that all the curves take a finite value at $R=0$. We found it necessary to sum up to 20 in the parameter m and to integrate up to 1000 in the variable k in order to obtain reliable plots. All the calculations were performed with computer programming, relying on a precision of 34 digits.

B. Spinor field

The asymptotic part of the energy is rewritten in the form

$$\mathcal{E}_{as}^{spin}=\frac{1}{12\pi R^2}[\beta^2 e_1(m_e R)+\beta^4 e_2(m_e R)], \tag{59}$$

where $e_1(x)$ and $e_2(x)$ are linear combinations of the integrals given in Eq. (54). Their asymptotic behavior for $x\to 0$ is

$$e_1(x)\sim 2+\mathcal{O}(x),$$

$$e_2(x)\sim -1+\mathcal{O}(x); \tag{60}$$

therefore $\mathcal{E}_{as}^{spin}$ is proportional to R^{-2} for small values of R. For $R\to\infty$ $\mathcal{E}_{as}^{spin}$ falls exponentially.

The finite part of the energy is also proportional to R^{-2} in the limit $R\to 0$ and to R^{-3} for $R\to\infty$. The plots of $\mathcal{E}_{as}^{spin}$, $\mathcal{E}_f^{spin}$, and of $\mathcal{E}_{ren}^{spin}$ are displayed below. The remarks made above for the graphics are also valid here.

V. CONCLUSIONS

In this paper we have carried out a complete calculation of the vacuum energy of two different fields in the background of a magnetic string with a delta function profile. The renormalized vacuum energy is given in terms of convergent integrals [Eqs. (36) and (37) for the scalar field, Eqs. (51) and (56) for the spinor field]. A first remark can be made about the vanishing of the heat-kernel coefficient A_2 in both scalar and spinor cases. This coefficient, contributing to $\mathcal{E}_{div}$, is not zero for a generic background potential. The vanishing of A_2 is also observed in a dielectric spherical shell with a squared profile in the dilute approximation [5]. It could be argued that more singular profiles possess less ultraviolet divergences than smooth profiles. This statement is also confirmed by the heat-kernel coefficients calculated in [12].

The dependence of the sign of the energy on the radius R of the string and on the potential strength β is nontrivial. The sign varies with the dimensions of the string. In the scalar case (Figs. 1 and 2) the energy is negative only for large values of the potential strength, while for β smaller than one, the energy shows a maximum. In the spinor case (Figs. 3 and 4) we have almost an opposite situation: in the region $R<1$ the energy shows a minimum for $\beta<1$, while it is positive for $\beta>1$. However, when R becomes large the vacuum energy shows the same behavior for the scalar and for the spinor field: it is negative for large β and positive for small β. The changing of the sign with the dimensions of the cavity cannot be simply ascribed to the peculiarity of the delta function potential. In facts, a similar behavior was observed in earlier works [21,8], where Dirichlet boundary conditions and bag boundary conditions were examined for a massive scalar field and a spinor field, respectively. We also note the strong dependence on the parameter β, which was not observed in [10], where a homogeneous field inside the flux tube was investigated. In fact a relevant result of our calculation is that the energy numerically shows a dependence on β^4 for large β. In [10] the contributions proportional to β^4 cancelled and the parts proportional to β^2 dominated the renormalized energy for large β. The proportionality $\mathcal{E}_{ren}\sim\beta^4$ opens the possibility that the inhomogeneity of the magnetic field could render the vacuum energy larger than the classical energy of the string (which depends only on β^2), for a sufficiently hard boundary. The total energy of the system could then be dominated by the quantum contribution. However in the model studied here, the profile of the potential contains a delta function and the classical energy is formally infinite. It would be interesting to study an inhomogeneous magnetic field which does not contain singularities in order to have a finite classical energy and pos-

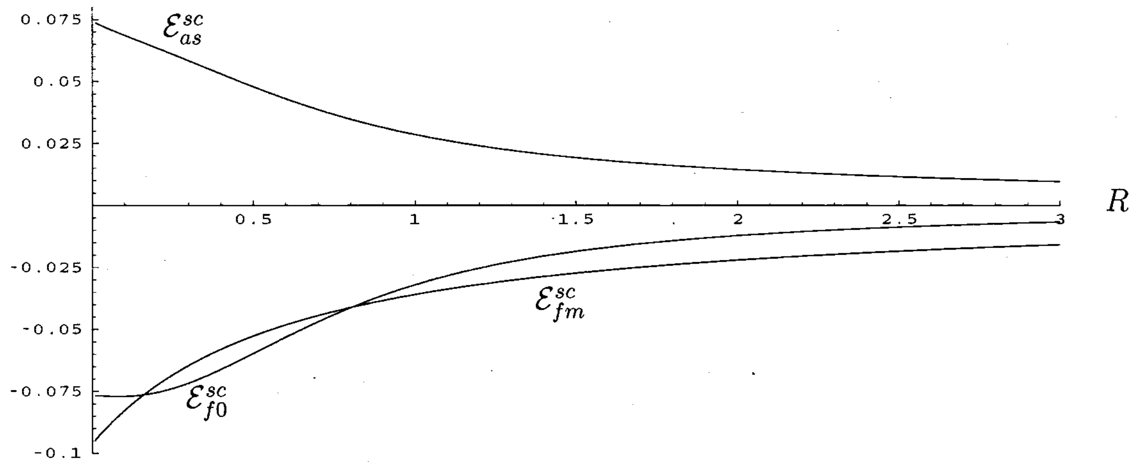

FIG. 1. Scalar field. $\mathcal{E}_{\mathrm{as}}^{\mathrm{sc}}$, $\mathcal{E}_{fm}^{\mathrm{sc}}$, and $\mathcal{E}_{f0}^{\mathrm{sc}}$ multiplied by $R^2 \cdot \beta^{-4}$, for $\beta = 2.2$.

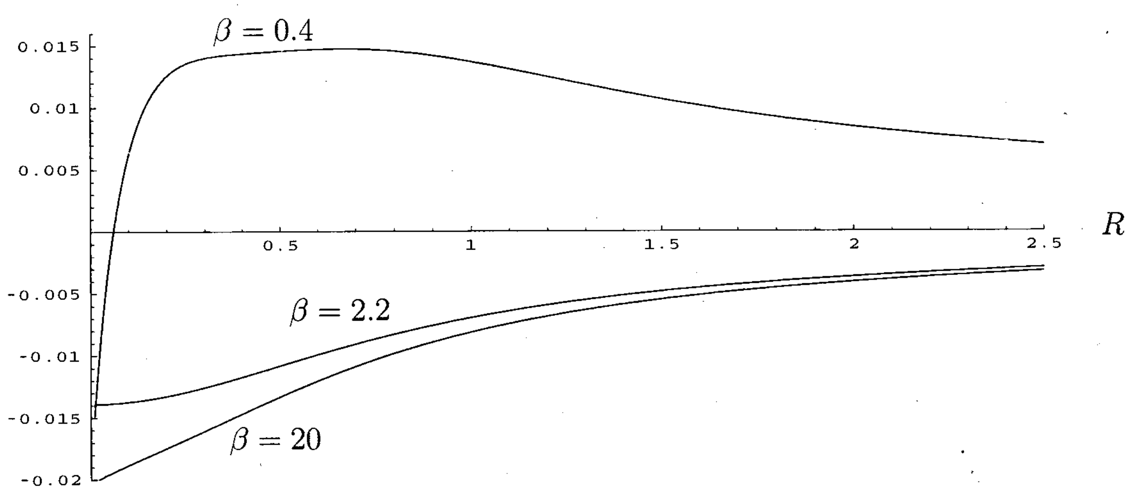

FIG. 2. Scalar field. The complete renormalized vacuum energy $\mathcal{E}_{\mathrm{ren}}^{\mathrm{sc}}(R)$ multiplied by $R^2 \cdot \beta^{-4}$, for different values of strength of the potential.

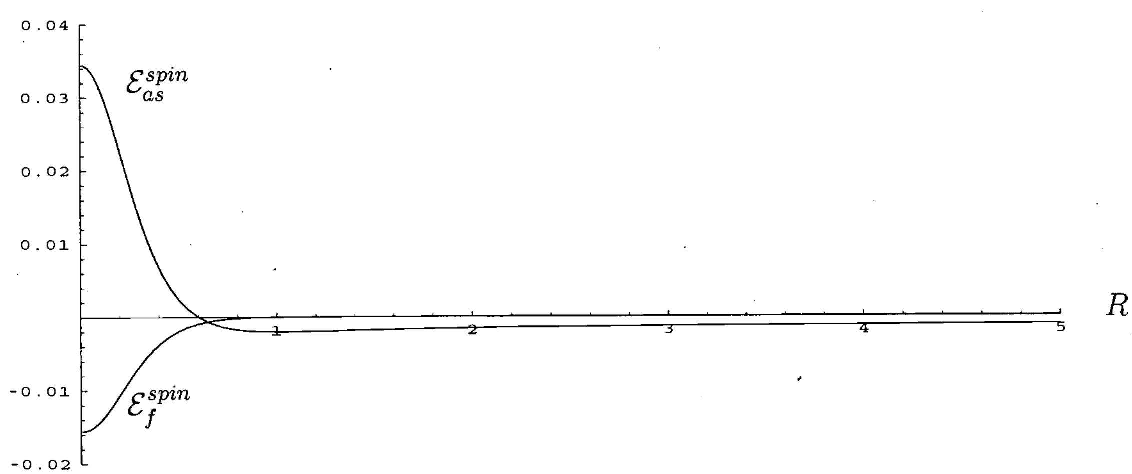

FIG. 3. Spinor field. The curves of the asymptotic and of the finite part of the energy multiplied by $R^2 \cdot \beta^{-4}$, for $\beta = 2.2$.

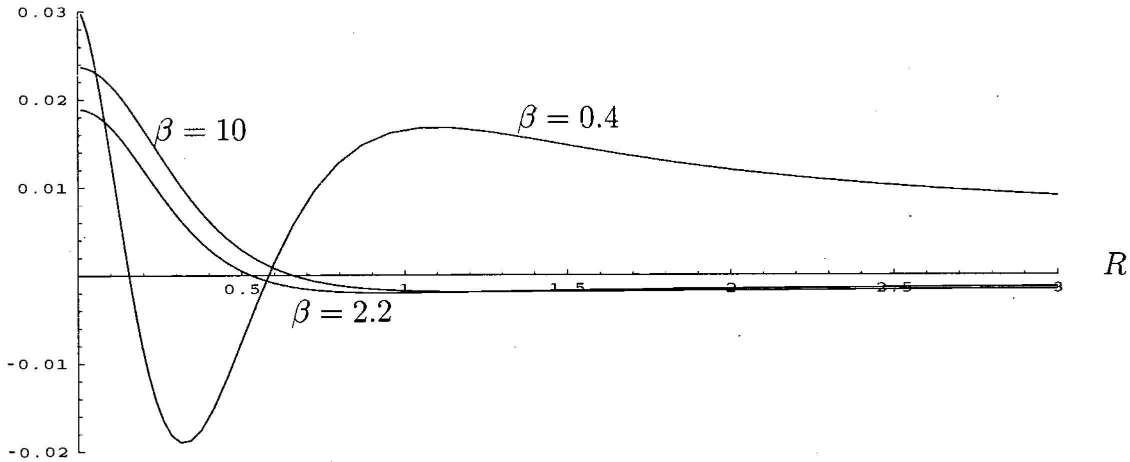

FIG. 4. Spinor field. The complete renormalized vacuum energy $\mathcal{E}^{\rm spin}(R)$ multiplied by $R^2 \cdot \beta^{-4}$, for different values of strength of the potential.

sibly a vacuum energy depending on β^4. Unfortunately with more realistic potentials the calculations become more difficult. A feasible model would be that of a cylindrical shell with finite thickness and with a profile given by a finite height box. In this case the Jost function would be expressed in terms of Bessel functions and hypergeometric functions. This problem is left for future investigation.

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APPENDIX A: EXPANSION OF THE MODIFIED BESSEL FUNCTIONS

The functions $S_\eta(n,\alpha,t)$ used in Eq. (23) are

$$S_\eta(-1,\alpha,t) = \eta t^{-1} + \eta \frac{1}{2}\ln\left(\frac{1-t}{1+t}\right),$$

$$S_\eta(0,\alpha,t) = \frac{1}{2}\ln t - \frac{\eta\alpha}{2}\ln\left(\frac{1+t}{1-t}\right),$$

$$S_\eta(1,\alpha,t) = -\frac{t}{24}(-3+12\alpha^2+12\alpha t+5t^2),$$

$$S_\eta(2,\alpha,t) = \frac{t^2}{48}[\eta 8\alpha^3 t + 12\alpha^2(-1+2t^2) + \alpha(-26t+30t^3)+3(1-6t^2+5t^4)],$$

$$S_\eta(3,\alpha,t) = \eta\frac{1}{128}\{[(25-104\alpha^2+16\alpha^4)t^3]/3 + 16\alpha(-7+4\alpha^2)t^4 - \eta(531/5-224\alpha^2 + 16\alpha^4)t^5 - [32\alpha(-33+8\alpha^2)t^6]/3 - \eta(-221+200\alpha^2)t^7 - 240\alpha t^8 - \eta(1105t^9)/9\}; \tag{A1}$$

where $t = 1/[1+(kR/m)^2]^{1/2}$ in the scalar case and $t=1/[1+(kR/\nu)^2]^{1/2}$ in the spinor case. The factor η is equal to 1 for the modified Bessel function I and to -1 for the modified Bessel function K.

APPENDIX B: CALCULATION OF THE INTEGRALS

The transformation of the sum in Eq. (51) into an integral has been done with the Abel–Plana formula half integer numbers (see [22], p. 31)

$$\sum_{m=0}^{\infty} F\left(m+\frac{1}{2}\right) = \int_0^\infty d\nu F(\nu) + \int_0^\infty \frac{d\nu}{1+e^{2\pi\nu}}\frac{F(i\nu)-F(-i\nu)}{i}. \tag{B1}$$

The following formulas, taken from [10], have been used for the integration over m and k in $\mathcal{E}^{\rm sc}_{\rm as1}$, $\mathcal{E}^{\rm sc}_{\rm as2}$, and in $\mathcal{E}^{\rm spin}_{\rm as(1)}$ [Eqs. (31), (32), and (52)]:

$$\int_0^\infty dm \int_{m_e}^\infty dk (k^2-m_e^2)^{1-s}\partial_k \frac{t^j}{m^n} = -\frac{m_e^{2-2s}}{2}\frac{\Gamma(2-s)\Gamma\left(\frac{1+j-n}{2}\right)\Gamma\left(s+\frac{n-3}{2}\right)}{(Rm_e)^{n-1}\Gamma(j/2)}, \tag{B2}$$

where $t=1/[1+(kR/m)^2]^{1/2}$ in the scalar case and $t=1/[1+(kR/\nu)^2]^{1/2}$ in the spinor case. The contributions $\mathcal{E}^{\rm sc}_{\rm as3}$ and $\mathcal{E}^{\rm spin}_{\rm as2}$ [Eqs. (33) and (53)] have been calculated by means of the formula

$$\int_{m_e}^{\infty} dk(k^2-m_e^2)^{1-2}\partial_k \frac{t^j}{m^n} = -m_e^{2-2s} \frac{\Gamma(2-s)\Gamma\left(s+\frac{j}{2}-1\right)m^{j-n}}{\Gamma\left(\frac{j}{2}\right)(Rm_e)^j\left[1+\left(\frac{m}{m_eR}\right)^2\right]^{s+(j/2)-1}}, \tag{B3}$$

and integrating several times by parts.

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